

On empty pentagons and hexagons in planar point sets*

Pavel Valtr[†]

Abstract

We give improved lower bounds on the minimum number of k -holes (empty convex k -gons) in a set of n points in general position in the plane, for $k = 5, 6$.

Keywords: Empty polygon, planar point set, empty hexagon, empty pentagon

1 Introduction

We say that a set P of points in the plane is in *general position* if it contains no three points on a line.

Let P be a set of n points in general position in the plane. A k -hole of P (sometimes also called empty convex k -gon or convex k -hole) is a set of vertices of a convex k -gon with vertices in P containing no other points of P .

Let $X_k(n)$ be the minimum number of k -holes in a set of n points in general position in the plane. Horton [7] proved that $X_k(n) = 0$ for any $k \geq 7$ and for any positive integer n . The following bounds on $X_k(n)$, $k = 3, 4, 5, 6$, are known (the letter H denotes the number of vertices of the convex hull of the point set):

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[†]Department of Applied Mathematics and Institute for Theoretical Computer Science (ITI), Charles University, Malostranské nám. 25, 182 00 Praha 1, Czech Republic

$$n^2 - \frac{43}{9}n + H + \frac{11}{9} \leq X_3(n) \leq 1.6195\dots n^2 + o(n^2),$$

$$\frac{n^2}{2} - \frac{55}{18}n + H - \frac{23}{9} \leq X_4(n) \leq 1.9396\dots n^2 + o(n^2),$$

$$\frac{2}{9}n - O(1) \leq X_5(n) \leq 1.0206\dots n^2 + o(n^2),$$

$$\frac{n}{463} - 1 \leq X_6(n) \leq 0.2005\dots n^2 + o(n^2).$$

The upper bounds were shown in [2], improving previous bounds of [9, 1, 11, 4]. The lower bounds for $k = 3, 4, 5$ can be found in an updated version of the conference paper [6], also improving lower bounds from several papers. The lower bound on $X_6(n)$ follows from a result of V. A. Koshelev [8]. In this paper we give the following improved lower bounds:

Theorem 1

$$\begin{aligned} X_5(n) &\geq n/2 - O(1), \\ X_6(n) &\geq n/229 - 4. \end{aligned}$$

After finishing our research, we have learned that a group of researchers including Oswin Aichholzer, Ruy Fabila-Monroy, Clemens Huebner, and Birgit Vogtenhuber has very recently obtained a better bound $X_5(n) \geq 3n/4 - o(n)$. Their result is not written yet. Their method does not seem to achieve our bound on $X_6(n)$ but it also gives slight improvements on the lower bounds on $X_3(n)$ and $X_4(n)$ mentioned above.

2 Proofs

To prove the first inequality in Theorem 1, it suffices to prove that if P is a set of $n > 20$ points in general position in the plane then P contains a subset P' of eight points such that P' and $P - P'$ can be separated by a line and at least four 5-holes of P intersect P' . Indeed, if this is true then we can repeatedly remove eight points of P' . Each removal decreases the number of points by 8 and the number of 5-holes by at least 4. Doing this as long as at least 21 points remain, we obtain the first inequality in Theorem 1.

Let P be a set of $n > 20$ points in general position in the plane. For two points x, y of P , we denote by $L(xy)$ the open halfplane to the left of the line xy (oriented from x to y). The complementary open halfplane is denoted by $R(xy)$. If $L(xy)$ contains exactly k points of P , then we say that the oriented segment xy is a k -edge of P .

Take a vertex a of the convex hull of P . Order the other points radially around a starting from the point on the convex hull clockwise from a . Let a' be the 12-th point in this order. Then aa' is an 11-edge. Since $X_5(10) > 0$ [5], $L(aa')$ contains a 5-hole, D , of P . In the rest of the proof, D is fixed but aa' may later denote other 11-edges.

The key part of the proof is to find an 11-edge bb' such that b is a vertex of D and the other four vertices of D lie in $L(bb')$. To do it, we clockwise rotate a line l starting from $l = aa'$ as follows. Initially we start to rotate l at the midpoint of the segment aa' . During the rotation, the center of rotation may change at any moment but the rotated line l cannot go over any point of P . We rotate as long as it is possible, until we reach a position $l = bb'$, where $b, b' \in P$, the point b was originally to the left of l and b' was originally to the right of l . Thus, $b \in L(aa') \cup \{a'\}$ and $b' \in R(aa') \cup \{a\}$. There are no points of P in the open wedges $R(aa') \cap L(bb')$ and $L(aa') \cap R(bb')$. The edge bb' is an 11-edge of P . We distinguish three cases:

Case 1: The segments aa' and bb' internally cross, thus a, a', b, b' are pairwise different.

Case 2: $b' = a$.

Case 3: $b = a'$.

Since D lies in $L(aa')$, it also lies in $L(bb') \cup \{b\}$. The point b may be a vertex of D in Cases 1 and 2. All other vertices of D lie in $L(bb')$. If b is not a vertex of D , then we rename the points b and b' by a and a' , respectively, and rotate a line l in the same way as above from the position $l = aa'$. We reach some new position $l = bb'$. Repeat this process until the point b coincides with one of the vertices of D . (Note that the line l cannot rotate outside of D forever, because $n > 20$.) Then we are in Case 1 or in Case 2, and the other four vertices of D lie in $L(aa') \cap L(bb')$. In Case 1 or 2, we consider the 12-point set $Q := (P \cap L(bb')) \cup \{b'\}$. Since $X_5(12) \geq 3$ [3], the set Q contains at least three 5-holes of P . Together with D , these are at least four 5-holes of P with vertices in the 13-point set $Q \cup \{b\} = P \cap \text{closure}(L(bb'))$. None of these 5-holes contains both b and b' . Therefore, we can take P' as the set of eight points of $L(bb')$ with largest distances to the line bb' . This finishes the proof of the first inequality in Theorem 1.

We remark without proof that a slightly better bound $(1/2 + c)n - \text{const}$ with $c > 0$ can be obtained by using the fact that any sufficiently large set P contains linearly many disjoint 6-holes.

The above proof can be generalized to give the more general theorem below. The theorem below together with $X_6(463) > 0$ (proved by V. A. Koshelev [8]) gives the second inequality in Theorem 1.

Theorem 2 *Suppose that $X_k(s-1) \geq 1$ and $X_k(s) \geq t$ for some positive integers k, s, t . Then $X_k(n) \geq \frac{t+1}{s-k+1}(n - (2s-2))$ for $n \geq 2s-2$.*

Proof. If P is a set of $n > 2s-2$ points then P contains an $(s-1)$ -edge aa' . Let D be a k -hole of P contained in $L(aa')$. Analogously as in the previous proof, we find two $(s-1)$ -edges aa' and bb' such that b is a vertex of D and D lies in $L(aa')$ and also in $L(bb') \cup \{b\}$. In Case 1 or 2, we consider the s -point set $Q := (P \cap L(bb')) \cup \{b'\}$. Since $X_k(s) \geq t$, the set Q contains at least t k -holes of P . Together with D , these are at least $t+1$ k -holes of P with vertices in the $s+1$ -point set $Q \cup \{b\} = P \cap \text{closure}(L(bb'))$. None of these k -holes contains both b and b' . Therefore, if we take P' as the set of $s-k+1$ points of $L(bb')$ with largest distances to the line bb' then removing the $s-k+1$ points of P' from P decreases the number of k -holes by at least $t+1$. Theorem 2 follows.

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